Trigonometry 3

1. Express $\sin 2\theta + \sqrt{3}\cos 2\theta$ in the form $R\sin(2\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence,

(i) find the maximum value of $\left(\sin 2\theta + \sqrt{3}\cos 2\theta\right)^2$ and the corresponding angle, θ at which the maximum value occurs.

(ii) Solve $\sin 2\theta + \sqrt{3}\cos 2\theta = 1$ for $0 \le \theta \le \pi$.

$\sin 2\theta + \sqrt{3}\cos 2\theta = 2\left(\frac{1}{2}\sin 2\theta + \frac{\sqrt{3}}{2}\cos 2\theta\right) = 2\left(\sin 2\theta \cos\frac{\pi}{3} + \cos 2\theta \sin\frac{\pi}{3}\right) = 2sin\left(2\theta + \frac{\pi}{3}\right)$

$= R\sin(2\theta + \alpha)$, where $R = 2 > 0$ and $0 < \alpha = \frac{\pi}{3} < \frac{\pi}{2}$

(i) $\left(\sin 2\theta + \sqrt{3}\cos 2\theta\right)^2 = 4sin^2\left(2\theta + \frac{\pi}{3}\right)$

Since $sin^2\left(2\theta + \frac{\pi}{3}\right) \le 1$,

Therefore, the maximum of $\left(\sin 2\theta + \sqrt{3}\cos 2\theta\right)^2$ is 4.

Maximum occurs when $sin^2\left(2\theta + \frac{\pi}{3}\right) = 1, or\ \ sin\left(2\theta + \frac{\pi}{3}\right) = \pm 1$

When $sin\left(2\theta + \frac{\pi}{3}\right) = 1, \quad 2\theta + \frac{\pi}{3} = 2n\pi + \frac{\pi}{2} \Longrightarrow \theta = n\pi + \frac{\pi}{12}$, where $n\epsilon\boldsymbol{Z}$.

When $sin\left(2\theta + \frac{\pi}{3}\right) = -1, \quad 2\theta + \frac{\pi}{3} = 2n\pi - \frac{\pi}{2} \Longrightarrow \theta = n\pi - \frac{5\pi}{12}$, where $n\epsilon\boldsymbol{Z}$.

(ii) $\sin 2\theta + \sqrt{3}\cos 2\theta = 1 \Longrightarrow 2sin\left(2\theta + \frac{\pi}{3}\right) = 1$

$sin\left(2\theta + \frac{\pi}{3}\right) = \frac{1}{2} \Longrightarrow 2\theta + \frac{\pi}{3} = 2n\pi + \frac{\pi}{6}\ or\ 2n\pi + \frac{5\pi}{6} \Longrightarrow \theta = n\pi - \frac{\pi}{12}\ or\ n\pi + \frac{\pi}{4}$

Since $0 \le \theta \le \pi,\ \theta = \frac{\pi}{4}, \frac{11\pi}{12}$.

2. Prove **(i)** $cos^{-1}(-x) = \pi - cos^{-1}x$ **(ii)** $\frac{1}{4}\pi + tan^{-1}x = tan^{-1}\left(\frac{1+x}{1-x}\right)$

If $cos^{-1}x = y,\ 0 \le y \le \pi$, then the principal values of $sin^{-1}x\epsilon\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(i) Method 1

$\cos[cos^{-1}(-x) + cos^{-1}x]$

$= \cos[cos^{-1}(-x)]\,cos[cos^{-1}x] - \sin[cos^{-1}(-x)]\,sin[cos^{-1}x]$

$= (-x)x - \sqrt{1-(-x)^2}\sqrt{1-x^2} = -1$

$\therefore cos^{-1}(-x) + cos^{-1}x = cos^{-1}(-1) = \pi \Longrightarrow cos^{-1}(-x) = \pi - cos^{-1}x$

Method 2

Let $f(x) = cos^{-1}(-x) + cos^{-1}x$

$$f'(x) = -\frac{1}{\sqrt{1-(-x)^2}}(-1) - \frac{1}{\sqrt{1-x^2}} = 0$$

Integrate, $f(x) = c$, a constant.

$cos^{-1}(-x) + cos^{-1}x = c$

Put $x = 1, cos^{-1}(-1) + cos^{-1}1 = c \Longrightarrow c = \pi$

Hence, $cos^{-1}(-x) + cos^{-1}x = \pi \Longrightarrow cos^{-1}(-x) = \pi - cos^{-1}x$

Method 3

$x\epsilon[-1,1] \Longrightarrow \exists_1 \theta\epsilon[0,\pi] s.t. \cos\theta = x \ and \ \cos(\pi-\theta) = -x$

$cos^{-1}(-x) + cos^{-1}x = cos^{-1}(\cos\theta) + cos^{-1}[\cos(\pi-\theta)] = \theta + (\pi-\theta) = \pi$

$\therefore cos^{-1}(-x) = \pi - cos^{-1}x$

(ii) $tan\left[tan^{-1}\left(\frac{1+x}{1-x}\right) - tan^{-1}x\right] = \frac{\left(\frac{1+x}{1-x}\right)-x}{1+\left(\frac{1+x}{1-x}\right)x} = 1 \Longrightarrow tan^{-1}\left(\frac{1+x}{1-x}\right) - tan^{-1}x = tan^{-1}1 = \frac{1}{4}\pi$

$\therefore \ \frac{1}{4}\pi + tan^{-1}x = tan^{-1}\left(\frac{1+x}{1-x}\right)$

3. Prove : $\frac{cos^4\theta+sin^4\theta}{cos^4\theta-sin^4\theta} = \frac{1}{2}(\cos 2\theta + \sec 2\theta)$

$$\frac{cos^4\theta+sin^4\theta}{cos^4\theta-sin^4\theta} = \frac{cos^4\theta+sin^4\theta}{(cos^2\theta-sin^2\theta)(cos^2\theta+sin^2\theta)} = \frac{cos^4\theta+sin^4\theta}{cos^2\theta-sin^2\theta} = \frac{\left(\frac{1+\cos 2\theta}{2}\right)^2+\left(\frac{1-\cos 2\theta}{2}\right)^2}{\cos 2\theta} = \frac{1}{2}\frac{\cos^2 2\theta+1}{\cos 2\theta}$$

$$= \frac{1}{2}(\cos 2\theta + \sec 2\theta)$$

4. Solve the equation $2\cos\theta + 5\sin\theta = 4$ by expressing in the form $cos\,(2\theta+\alpha)$, where $0° \leq \theta \leq 360°$

$2\cos\theta + 5\sin\theta = 4$

Squaring, $4\cos^2\theta + 20\sin\theta\cos\theta + 25\sin^2\theta = 16$

$\Longrightarrow 4\cos^2\theta + 20\sin\theta\cos\theta + 25(1-\cos^2\theta) = 16$

$\Longrightarrow 21\cos^2\theta - 20\sin\theta\cos\theta = 9$

$\Longrightarrow 42\cos^2\theta - 40\sin\theta\cos\theta = 18$

$\Longrightarrow 21(2\cos^2\theta - 1) - 20(2\sin\theta\cos\theta) = -3$

$\Longrightarrow 21\cos 2\theta - 20\sin 2\theta = -3$

$$\Longrightarrow \frac{21}{\sqrt{21^2+20^2}}\cos 2\theta - \frac{20}{\sqrt{21^2+20^2}}\sin 2\theta = -\frac{3}{\sqrt{21^2+20^2}}$$

$$\Longrightarrow \frac{21}{29}\cos 2\theta - \frac{20}{29}\sin 2\theta = -\frac{3}{29}$$

Put $\cos\alpha = \frac{21}{29}$, $\sin\alpha = \frac{20}{29}$, $\alpha = \cos^{-1}\left(\frac{21}{29}\right) \approx 43.6028189727036°$

We get $\cos 2\theta \cos\alpha - \sin 2\theta \sin\alpha = -\frac{3}{29}$

$\cos(2\theta + \alpha) = -\frac{3}{29}$

$2\theta + \alpha = 360°n \pm (180° - 84.0622275483946°)$, where n is an integer.

$\theta = (180°n \pm 47.9688862258027°) - \frac{43.6028189727036°}{2}$

$= (180°n \pm 47.9688862258027°) - 21.8014094863518°$

Since $0° \leq \theta \leq 360°, \theta = (180°(0) + 47.9688862258027°) - 21.8014094863518°$

$\theta \approx 26.1674767394509°$

Or $\theta = (180°(1) + 47.9688862258027°) - 21.8014094863518°$

$\theta \approx 206.1674767394509°$

Note that squaring an equation may create roots, since

$2\cos\theta + 5\sin\theta \approx 2\cos 206.1674767394509° + 5\sin 206.1674767394509° = -4$

Therefore $\theta \approx 206.1674767394509°$ is rejected.

$\therefore \theta \approx \mathbf{26.1674767394509}°$

5. Prove that : $\frac{d^n}{dx^n}\sin x = sin\left(x + \frac{n\pi}{2}\right)$

Let $P(n)$ be the proposition: $\frac{d^n}{dx^n}\sin x = sin\left(x + \frac{n\pi}{2}\right)$.

We like to use the Principle of Mathematical Induction to prove that $P(n)$ is true $\forall n \epsilon \mathbb{N}$.

For $P(1)$, $\frac{d}{dx}\sin x = \cos x = sin\left(x + \frac{\pi}{2}\right)$. $\therefore P(1)$ is true.

Assume $P(k)$ is true for some , that is, $\frac{d^k}{dx^k}\sin x = sin\left(x + \frac{k\pi}{2}\right)$ ... (∗)

For $P(k+1)$, $\frac{d^{k+1}}{dx^{k+1}}\sin x = \frac{d}{dx}\left[\frac{d^k}{dx^k}\sin x\right] = \frac{d}{dx}\left[sin\left(x + \frac{k\pi}{2}\right)\right]$, by (*)

$$= cos\left(x + \frac{k\pi}{2}\right) = sin\left[\frac{\pi}{2} + \left(x + \frac{k\pi}{2}\right)\right] = sin\left[x + \frac{(k+1)\pi}{2}\right]$$

$\therefore P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true $\forall n \epsilon \mathbb{N}$.

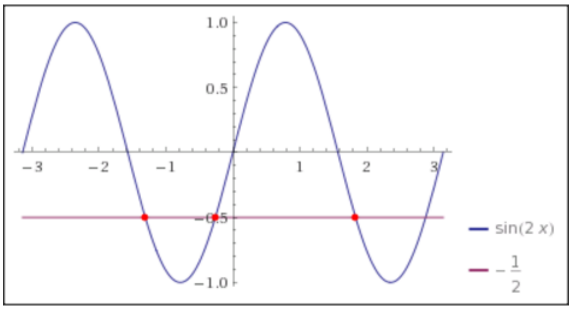
6. **(i)** Find the set of values of x in the interval $-\pi < x < \pi$ such that $|\sin 2x| > \frac{1}{2}$.

(ii) From **(i)**, find the set of values of x in the interval $-\pi < x < \pi$ such that $|\sin 2x| > \frac{1}{2}$

(i) $|\sin 2x| > \frac{1}{2} \Longrightarrow -\frac{1}{2} > \sin 2x \quad or \quad \sin 2x > \frac{1}{2}$

For $\sin 2x = -\frac{1}{2}$, $2x = -\pi + \frac{\pi}{6}, -\frac{\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$x = -\frac{5}{12}\pi, -\frac{1}{12}\pi, \frac{7}{12}\pi, \frac{11}{12}\pi$, where $-\pi < x < \pi$.

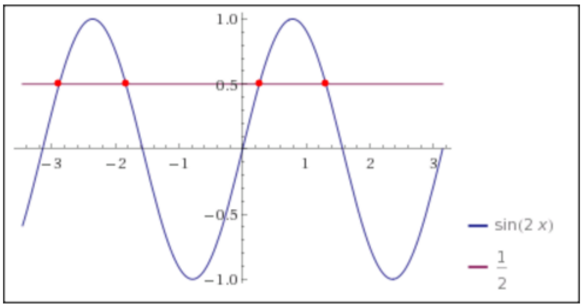


Hence for $-\frac{1}{2} > \sin 2x$,

$-\frac{5}{12}\pi < x < -\frac{1}{12}\pi, \frac{7}{12}\pi < x < \frac{11}{12}\pi,$ ... (1)

For $\sin 2x = \frac{1}{2}$, $2x = -2\pi + \frac{\pi}{6}, -\pi - \frac{\pi}{6},, \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$x = -\frac{11}{12}\pi, -\frac{7}{12}\pi, \frac{1}{12}\pi, \frac{5}{12}\pi$, where $-\pi < x < \pi$.



Hence for $\sin 2x > \frac{1}{2}$,

$-\frac{11}{12}\pi < x < -\frac{7}{12}\pi$, $\frac{1}{12}\pi < x < \frac{5}{12}\pi$ ... (2)

For $|\sin 2x| < \frac{1}{2}$, the solution is the union of (1) and (2),

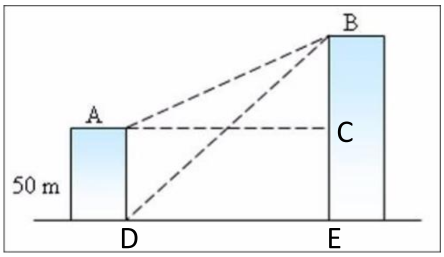
$-\frac{11}{12}\pi < x < -\frac{7}{12}\pi$, $-\frac{5}{12}\pi < x < -\frac{1}{12}\pi, \frac{1}{12}\pi < x < \frac{5}{12}\pi$

(ii) $-\pi < x < -\frac{11}{12}\pi$, $-\frac{7}{12}\pi < x < -\frac{5}{12}\pi$, $-\frac{1}{12}\pi < x < \frac{1}{12}\pi$, $\frac{5}{12}\pi < x < \frac{7}{12}\pi$, $\frac{11}{12}\pi < x < \pi$

7. Referring to the diagram, building A is measured with 50 m in height. The angle from the base of the building A to the highest point of building B is measured as 62°. The angle formed from the rooftop of building A right to the highest point of building B is 59°. What is the distance that keeps the two buildings apart?

Method 1

Let $h = BC, d = DE = AC$. $\angle BAC = 59°, \angle BDE = 62°$.



$\tan 59° = \frac{h}{d} \Longrightarrow h = d \tan 59°$ ... (1)

$\tan 62° = \frac{h+50}{d} \Longrightarrow h + 50 = d \tan 62°$ ... (2)

(1) ↓ (2), $d \tan 59° + 50 = d \tan 62°$

$\therefore d = \frac{50}{\tan 62° - \tan 59°} \approx \frac{50}{1.880726465 - 1.664279482} \approx 231.00345\ (m)$

Method 2

$\angle ABD = (90° - 59°) - (90° - 62°) = 3°$, $\angle ABD = 90° + 59° = 149°$

By Sine Law, $BD = \frac{50}{\sin 3°} \times \sin 149° \approx \frac{50}{0.05233595624} \times 0.5150380749 \approx 492.049932687$

$\therefore d = DE = 492.049932687 \cos 62° \approx 231.00345\ (m)$

Method 3

$\angle ABD = (90° - 59°) - (90° - 62°) = 3°$

$AB = d \sec 59°, BD = d \sec 62°$

Apply Cosine Law to ΔABD , $50^2 = (d \sec 59°)^2 + (d \sec 62°)^2 - 2(d \sec 59°)(d \sec 62°) \cos 3°$

$d^2 = \frac{50^2}{(\sec 59°)^2 + (\sec 62°)^2 - 2(\sec 59°)(\sec 62°) \cos 3°} \approx 53362.594$, $\therefore d \approx 231.00345\ (m)$

8. It is given that $f(x) = 11\, cos^2 x - 6 \sin x \cos x + 3\, sin^2 x$.

(a) Express $f(x)$ in the form of $a\, cos\, 2x + b \sin 2x + c$, where a, b and c are constants.

(b) Show that $f(x)$ can be expressed in the form of $rcos\,(2x + \alpha) + c$, where r and c are constants and $\tan \alpha = \frac{3}{4}$.

(c) Find the maximum and minimum values of the express $\frac{1}{f(x)}$.

(d) Find the values of x between $0°$ and $180°$ such that $f(x) = 8$.

(e) Find the set of values of x in the interval $0° \le x \le 180°$ such that $2 \le f(x) \le 4.5$.

(a) $f(x) = 11\, cos^2 x - 6 \sin x \cos x + 3\, sin^2 x = \frac{11}{2}(1 + \cos 2x) - 3 \sin 2x + \frac{3}{2}(1 - \cos 2x)$

$= 4\, cos\, 2x - 3 \sin 2x + 7$, $a = 4, b = -3, c = 7$.

(b) $f(x) = 5\left(\frac{4}{5}\, cos\, 2x - \frac{3}{5} \sin 2x\right) + 7$

$= 5(\, cos\, 2x \cos \alpha - \sin 2x \sin \alpha) + 7$, where $\tan \alpha = \frac{3}{4}$.

$= 5 \cos(2x + \alpha) + 7$.

(c) $-1 \le \cos(2x + \alpha) \le 1 \Rightarrow -5 \le 5 \cos(2x + \alpha) \le 5 \Rightarrow 2 \le 5 \cos(2x + \alpha) + 7 \le 12$

$\Rightarrow \frac{1}{12} \le \frac{1}{5 \cos(2x+\alpha)+7} \le \frac{1}{2}$. Max of $\frac{1}{f(x)} = \frac{1}{2}$ and min. of $\frac{1}{f(x)} = \frac{1}{12}$.

(d) $f(x) = 8 \Rightarrow 5 \cos(2x + \alpha) + 7 = 8 \Rightarrow \cos(2x + \alpha) = \frac{1}{5} \Rightarrow 2x + \alpha = 360°n \pm 78.46304°$

$\Rightarrow 2x = 360°n \pm 78.46304° - 36.86990° \Rightarrow x = 180°n \pm 39.23152° - 18.4349°$

When $x = 39.23152° - 18.4349° \approx 20.797°$

or $x = 180° - 39.23152° - 18.4349° = 122.334°$

(e) $2 \le f(x) \le 4.5 \Rightarrow 2 \le 5 \cos(2x + \alpha) + 7 \le 4.5 \Rightarrow -5 \le 5 \cos(2x + \alpha) \le -2.5$

$\Rightarrow -1 \le \cos(2x + \alpha) \le -\frac{1}{2} \Rightarrow \cos(2x + \alpha) \le -\frac{1}{2} \Rightarrow 120° \le 2x + 78.46304° \le 240°$

$\Rightarrow 20.768° \le X \le 95.768°$

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